

Quantum Boltzmann equation for a mobile impurity in a degenerate Tonks-Girardeau gas

O. Gamayun^{1,2}

¹ *Lancaster University, Physics Department, Lancaster LA1 4YB, UK and*

² *Bogolyubov Institute for Theoretical Physics, 14-b Metrolohichna str., Kyiv 03680, Ukraine*

We investigate large time asymptotical behavior of a mobile impurity immersed in a degenerate Tonks-Girardeau gas. We derive correct weak-coupling kinetic equation valid for arbitrary ratio of masses of gas and impurity particles. When gas particles are either lighter or heavier than the impurity we find that our theory is equivalent to the Boltzmann theory with collision integral calculated via Fermi Golden Rule. On the contrary, in the equal masses case Fermi Golden Rule treatment gives false results due to not accounting for multiple coherent scattering events. The latter are treated by the resummation of ladder diagrams, which leads to a new kinetic equation. Asymptotic momentum of the impurity produced from this equation coincides with the result obtained by means of Bethe Ansatz.

I. INTRODUCTION

The propagation of impurities in quantum liquids and gases always attracted a lot of attention of researchers [1–3]. Recently this interest revived with a focus on one-dimensional (1D) systems. Partially it is due to tremendous experimental progress in fabricating and manipulating ultracold atomic gases [4], which allowed to create and manipulate a single impurity state in a 1D host gas of bosons [5, 6] and address to its non-equilibrium dynamics [7–11].

Theoretical interest for impurity propagation in 1D systems is due to rich variety of unusual properties of the host liquids [12, 13]. One of the prominent feature is a substantial modification of the superfluidity (understood as the absence of friction force) in 1D liquids [14]. Another intriguing phenomenon is quasi-Bloch oscillations, which comprises in impurity's momentum oscillations in the presence of external force, this way, resembling the Bloch oscillations in an ideal crystal [15–17]. The most puzzling phenomena, though, concerning impurity motion is quantum flutter phenomenon discovered in Ref. [18] and further explored in Ref. [19]. Authors considered supersonic impurity injected into a one-dimensional gas of hardcore bosons, also known as the Tonks-Girardeau (TG) gas. This gas is equivalent to the free-fermion system [20]. At zero temperature host fermions form a Fermi sea in which impurity propagation is considered (more general host was considered in [19]). Numerical analysis in [18, 19] clearly shows that large time asymptotic of average momentum exhibits oscillations around some non-zero value. These results suggest existence of an asymptotical steady state with a non-vanishing momentum of the impurity. The nature of this state was analyzed in [21, 22] where a complete analytical theory for description of such state has been developed. The dependence of the asymptotic momentum p_∞ on initial momentum p_0 has been calculated in weak-coupling regime. From the kinetic arguments it is clear that if initial momentum is less than some critical $|p_0| < q_0$ then even a single act of scattering is prohibited by classical conservation

laws and asymptotic momentum of the impurity coincides with initial, $p_\infty = p_0$. This could be considered as Landau-like criteria (see also Ref. [23] for taking into account interaction and getting non-perturbative bounds on p_∞). If initial momentum is larger then q_0 then after several scattering events impurity's momentum drops below q_0 and further scattering stops. To calculate this asymptotical value semi-classical Boltzmann theory was invoked in Ref. [21]. The Boltzmann theory treatment heavily relies on the assumption that all the dynamics reduces to a sequence of a pairwise collisions. This is applicable when impurity mass is not equal to the host particle mass (non-equal masses case). On the other hand, for equal masses, impurity momentum drops below q_0 after the first scattering forming a hole in a Fermi sea. And, unlike to the non-integrable case, velocity of the hole equals to the velocity of the particle and they may experience multiple coherent scattering processes leading to a resonant interaction between the impurity and the host. When interaction between the impurity and the host is point-like, the equal masses case is integrable by means of coordinate Bethe Ansatz [24]. This technique was used in [21] and it was obtained that in the vanishing coupling constant limit asymptotic momentum acquire following non-trivial value:

$$p_\infty = p_0 - \theta(|p_0| - k_F) \frac{p_0^2 - k_F^2}{2k_F} \ln \frac{p_0 + k_F}{p_0 - k_F}. \quad (1)$$

Here k_F is Fermi momentum, which in this case coincides with q_0 . Note that same momentum calculated within Boltzmann theory approach would be:

$$p_\infty^B = p_0 - 2k_F \theta(|p_0| - k_F) \left(\ln \frac{p_0 + k_F}{p_0 - k_F} \right)^{-1}. \quad (2)$$

This clearly shows that even at vanishingly small coupling constant accounting for multiple scattering processes provides finite non-perturbative corrections to the final result.

In this manuscript we analyze this phenomenon in a systematic way. We derive kinetic equation that describes impurity's momentum probability distribution

from Schwinger-Dyson equation in Keldysh formalism. To solve this equation and find asymptotic distribution we have to specify diagrams used in impurity self-energy Σ (collision integrals). Standard Boltzmann theory corresponds to the Σ in the form of the single bubble diagram, which is the lowest order expansion in coupling constant. This approximation is equivalent to the Boltzmann equation with probability transitions computed by Fermi Golden Rule. Such approach works good in the non-equal masses case meaning that taking into account higher orders in Σ results to higher order corrections in the final answer for asymptotic probability distribution. At equal masses this is no longer true. Namely, asymptotic distribution calculated from bubble diagram at equal masses for initial impurity's momentum $p_0 > k_F$ is found to be:

$$[\omega_p^B]^\infty = \frac{1}{Z_{p_0}^B} \frac{\theta(k_F - |p|)}{p_0 - p}, \quad (3)$$

here $Z_{p_0}^B$ is an appropriate normalization constant. This result can be used to reproduce answer (2). However, if one takes into account, ladder diagrams and perform effective resummation in Σ , one will find that result for asymptotic distribution is drastically changed even in leading order, providing

$$\omega_p^\infty = \frac{1}{Z_{p_0}} \frac{\theta(k_F - |p|)}{(p_0 - p)^2}, \quad (4)$$

which leads to the correct result (1) obtained by the means of Bethe-Ansatz solution. If the mass ratio is far enough from unity we find that ladder effects are suppressed and Boltzmann theory is applicable. Also we are able to write down effective Boltzmann-like equation at the equal masses case replacing transitions rates obtained by Fermi Golden Rule by those calculated from ladder diagrams. Using this equation we consider dynamics of the external force applied to the impurity. Such a considerations are usually extremely difficult in integrable systems and few analytical results are obtained so far. It is straightforward to include finite temperature and trap potential in our approach, but we postpone this to separate consideration.

The plan of the paper is the following: in the next section we describe physical system and introduce main notations. In Sec. III we derive general kinetic equation to describe impurity and present results for kinetic equation with specific ladder contributions to the impurity's self-energy. In Sec. IV we determine expression for Green function, analyze asymptotic distributions and obtain main results of our paper, in particular, Eq. (4). Section (V) is devoted to the consideration of equal mass system with applied constant force. Finally, short summary and discussion are present in Sec. (VI).

II. PHYSICAL SYSTEM AND GENERAL SETUP

As it was announced in introduction, we consider impurity particle immersed in Tonks-Girardeau gas. This gas is equivalent to the free fermions [20] and we will use this fermionic descriptions of the host. The Hamiltonian of our system in the second quantization language reads:

$$H = \sum_p \epsilon_p a_p^\dagger a_p + \sum_p E_p b_p^\dagger b_p + g \sum_{p,q,s} b_{p-q}^\dagger b_p a_{s+q}^\dagger a_s \quad (5)$$

Here operators a_p corresponds to the host fermions and b_p to the impurity. The Hilbert space for impurity is reduced to be one-particle, i.e. it consists of vacuum state $|0\rangle$ and linear combinations of the one-particle states $b_p^\dagger|0\rangle$ only. In this case it is not necessary to specify the statistics of the operators b_p . But for certainty, whenever needed, we will assume that impurity is fermion presuming that all effects of statistics will cancel out in final answers. The initial state of the whole system $|\Omega\rangle$, for the sake of simplicity, is taken to be a tensor product of the impurity at given momentum p_0 and host particles in the Fermi sea state that is defined by momentum k_F : $|\Omega\rangle = b_{p_0}^\dagger|0\rangle \otimes |FS\rangle$. The spectrum of particles are assumed to be $\epsilon_p = p^2/2m_h$ and $E_p = p^2/2m_i$, though some results presented below are valid for arbitrary spectrum. System is assumed to have periodic boundary conditions with period L , and we denote $\sum_p = L/2\pi \int dp$, and set $\hbar = 1$. The strength of the interaction is characterized by the following dimensionless coupling constant $\gamma \equiv (gL/2\pi)m_h/k_F$ and all final answers depend on γ only. Nevertheless, we keep g for convenience in intermediate calculations. It is natural to measure all the momenta in units of k_F and all masses in the mass of the host particle m_h . From now on we set $k_F = 1$ and $m_h = 1$. So impurity mass is now given by ratio $\eta = m_i/m_h$ and $\gamma = gL/2\pi$.

The long-time evolution from the the initial state $|\Omega\rangle$ is our primary concern. In particular we would like to know impurity's momentum distribution probability defined as:

$$\omega_p(t) = \langle \Omega | e^{itH} b_p^\dagger b_p e^{-itH} | \Omega \rangle. \quad (6)$$

The initial distribution is given by the formula:

$$\omega_p(0) = \delta_{p,p_0}. \quad (7)$$

The most natural approach for evaluating (6) is to use Keldysh formalism technique [25]. Following standard procedures [26, 27], we introduce operators α_p and β_p identical to a_p and b_p , which however have specific time ordering (live on a different contour). "Greek" operators precede "Latin" operators at any values of time, which runs from 0 to ∞ and are anti-time-ordered among themselves, while "Latin" operators are time-ordered. We perform Keldysh rotation by means of the following matrix

ces:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \tilde{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad (8)$$

introducing new variables

$$\chi_k = U \begin{pmatrix} b_k \\ \beta_k \end{pmatrix}, \quad \bar{\chi}_k = \tilde{U} \begin{pmatrix} b_k^+ \\ \beta_k^+ \end{pmatrix}, \quad (9)$$

$$\psi_k = U \begin{pmatrix} a_k \\ \alpha_k \end{pmatrix}, \quad \bar{\psi}_k = \tilde{U} \begin{pmatrix} a_k^+ \\ \alpha_k^+ \end{pmatrix}. \quad (10)$$

Interaction term takes following form:

$$H_{\text{int}} = \frac{g}{2} \sum_{a=0,1} \sum_{k,q,s} \bar{\chi}_{k-q} \sigma_{1-a} \chi_k \bar{\psi}_{s+q} \sigma_a \psi_s, \quad (11)$$

with

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (12)$$

The bare impurity Green function takes following form:

$$\langle \Omega | T \chi_k(t_1) \otimes \bar{\chi}_p(t_2) | \Omega \rangle \equiv \delta_{kp} \mathbf{G}_p^{(0)} e^{-i(t_1-t_2)E_p}, \quad (13)$$

where:

$$\mathbf{G}_p^{(0)} = \begin{pmatrix} G_p^+ & G_p^K \\ 0 & -G_p^- \end{pmatrix} = \begin{pmatrix} \theta(t_1-t_2) & 1-2\delta_{p,p_0} \\ 0 & -\theta(t_2-t_1) \end{pmatrix} \quad (14)$$

is the quantity that we will further refer as Green function. Our definition differs from the usual one by prefactor $e^{-i(t_1-t_2)E_p}$, which in momentum space corresponds to the shifting the energy shell to zero. Analogous Green function for host fermions reads:

$$\mathbb{F}_p = \begin{pmatrix} \theta(t_1-t_2) & \text{sgn}(|p|-k_F) \\ 0 & -\theta(t_2-t_1) \end{pmatrix} \quad (15)$$

We see that initial conditions enters through the Keldysh part of Green function (the upper right corner element). Therefore, it is useful to combine diagonal elements, introducing Feynman Green function:

$$G_p(t_1, t_2) = \theta(t_1-t_2) G_p^+(t_1, t_2) + \theta(t_2-t_1) G_p^-(t_1, t_2). \quad (16)$$

Further, it is also useful to consider the following combination instead of G_p^K :

$$W_p(t_1, t_2) = \frac{G_p(t_1, t_2) - G_p^K(t_1, t_2)}{2}. \quad (17)$$

The probability distribution is given by the following formula (6) as:

$$\omega_p(t) = W_p(t, t). \quad (18)$$

therefore, we will refer to quantity (17) as generalized probability distribution. It is of order $1/L$ contrary to Feynman Green Function (16) which is of order 1:

$$\frac{1}{L} \sim W_k \ll G_k \sim 1. \quad (19)$$

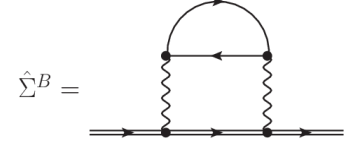


FIG. 1. The diagram for self-energy that corresponds to the Boltzmann approximation

One can show that because of such 'separation of scales' Feynman Green function remains translational invariant in limit $L \rightarrow \infty$ even though it is not a vacuum average. This is easily understood because one-particle contribution of the impurity produces only $1/L$ effect compared to vacuum state:

$$G_k(t_1, t_2) = G_k(t_1 - t_2) + O(1/L) \quad (20)$$

This can be considered as low density approximation, which is absolutely applicable here since we are dealing with a single impurity and thermodynamically large amount of host particles. The generalized probability distribution, however, retains essential dependence on both times. This distribution satisfies Quantum Boltzmann Equation which we derive in next section.

III. QUANTUM BOLTZMANN EQUATION

Let us denote self-energy of particle, which is given by all appropriate one-particle irreducible diagrams, as $\hat{\Sigma}$. The full Green function (14) is given by:

$$[\mathbf{G}]^{-1} = [\mathbf{G}^{(0)}]^{-1} - \hat{\Sigma} \quad (21)$$

The self-energy maintains the same matrix structure as Green function [26]:

$$\hat{\Sigma}_p(t_1, t_2) = \begin{pmatrix} \Sigma_p^+(t_1, t_2) & \Sigma_p^K(t_1, t_2) \\ 0 & -\Sigma_p^-(t_1, t_2) \end{pmatrix} \quad (22)$$

In the same manner as for Green function we may introduce following notations:

$$\Sigma_p(t_1, t_2) = \theta(t_1-t_2) \Sigma_p^+(t_1, t_2) + \theta(t_2-t_1) \Sigma_p^-(t_1, t_2) \quad (23)$$

$$\sigma_p(t_1, t_2) = \frac{\Sigma_p^K(t_1, t_2) - \Sigma_p(t_1, t_2)}{2} \quad (24)$$

Analogously as for Green functions we have $\Sigma_p \sim 1$ and $\sigma_p \sim 1/L$. In $L \rightarrow \infty$ limit Σ_p is translational invariant $\Sigma_p(t_1, t_2) = \Sigma_p(t_1 - t_2)$ so equation (21) transforms into system on two integral equations:

$$G_p(\tau) = 1 + \int_0^\tau dt \int_t^\tau dt_1 \Sigma_p(t_1 - t) G_p(t) \quad (25)$$

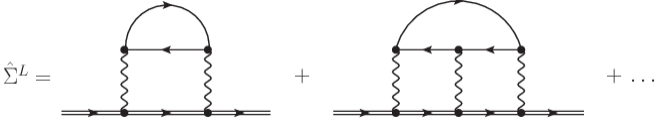


FIG. 2. The ladder diagram for self-energy that takes into account multiple impurity hole scattering events

$$W_p(\tau_1, \tau_2) = \omega_p^{(0)} G_p(\tau_1) G_p^*(\tau_2) + \int_0^{\tau_1} dt_1 \int_0^{\tau_2} dt_2 G_p(\tau_1 - t_1) \sigma_p(t_1, t_2) G_p^*(\tau_2 - t_2), \quad (26)$$

here $\omega_p^{(0)} = \delta_{p,p_0}$, * means complex conjugation. For the derivation of this equations the fact that $G_p(-\tau) = G_p^*(\tau)$ was used. We note that Eqs. (25) and (26) could be self-consistently considered for positive times only. Therefore we may apply Laplace transformation to these equation and obtain:

$$G_p(\lambda)^{-1} = \lambda - \Sigma_p(\lambda) \quad (27)$$

$$W_p(\lambda_1, \lambda_2) = G_p(\lambda_1) \left(\omega_p^{(0)} + \sigma_p(\lambda_1, \lambda_2) \right) G_p^*(\lambda_2) \quad (28)$$

One can easily note that in leading $1/L$ order Σ_p does not depends on W_p while σ_p depends in linear way. Therefore, Eqs. (25) and (27) are self-consistent and actually describe vacuum Green function. Once this function is found then Eqs. (26) and (28) present linear integral equation on generalized probability distribution, which

we will call them Quantum Boltzmann Equation (QBE) in time and λ space, respectively. The fact that QBE is linear is the essence of 'low-density approximation', which as was discussed above is exact in thermodynamical limit ($L \rightarrow \infty$).

Eqs. (27),(28) are valid for any systems in any dimensions. But to make them comprehensive we have to specify self-energy pertinent for our system. To do this we will use diagrammatic approach, which seems the most suitable in the perturbative limit. Notations in all diagrams are as follows: wiggled line corresponds to the interaction (11), simple line corresponds to host propagators (15) and double line corresponds to the impurity propagators (14). Because of one-particle Hilbert space each diagram can contain no more than one double line. In the leading order in coupling constant $\hat{\Sigma}$ is given by the diagram at Fig. 1.

This diagram describes act of the single scattering of the impurity on the host particle and leads to the Boltzmann theory based on semi-classical Fermi Golden Rule. However, as we point out in Introduction, for equal masses even from purely kinematic reasons one might expect that further scattering of the impurity on the hole are significant. Therefore we include in consideration so-called ladder diagrams presented at Fig. 2 even for arbitrary η .

We calculate $\hat{\Sigma}_p$ from the corresponding diagrams and then determine Σ_p and σ_p using definitions (23), (24). For the ladder diagrams (Fig. 2) result reads:

$$\Sigma_p^L(\lambda) = (ig)^2 \sum_{|k|>1} \frac{\sum_{|q|<1} G_{p-k+q}(\lambda + iE_{kq})}{1 - ig \sum_{|q|<1} G_{p-k+q}(\lambda + iE_{kq})} \quad (29)$$

$$\sigma_p^L(\lambda_1, \lambda_2) = g^2 \sum_{|s|<1, |k|>1} \frac{W_{p-s+k}(\lambda_1 + iE_{sk}, \lambda_2 - iE_{sk})}{\left(1 - ig \sum_{|q|<1} G_{p-s+q}(\lambda_1 + iE_{sq}) \right) \left(1 + ig \sum_{|q|<1} G_{p-s+q}^*(\lambda_2 - iE_{sq}) \right)} \quad (30)$$

where $E_{q_i q}$ is transferred energy:

$$E_{q_i q} = E_{p-q_i+q} - E_p + \epsilon_{q_i} - \epsilon_q \quad (31)$$

which for quadratic dispersions is equal to:

$$E_{q_i q} = \frac{q_i - q}{\eta} \left(q_i \frac{\eta + 1}{2} + q \frac{\eta - 1}{2} - p \right). \quad (32)$$

It retains dependence on p which is incoming impurity's momentum.

The Boltzmann diagrams (Fig. 1) corresponds to the lowest g orders that come from the ladder diagrams, namely:

$$\Sigma_p^B(\lambda) = (ig)^2 \sum_{|k|>1, |q|<1} G_{p-k+q}(\lambda + iE_{kq}) \quad (33)$$

$$\sigma_p^B(\lambda_1, \lambda_2) = g^2 \sum_{|s|<1, |k|>1} W_{p-s+k}(\lambda_1 + iE_{sk}, \lambda_2 - iE_{sk}) \quad (34)$$

At this point it is worth to emphasize the consistency of our approach regarding the sum rule

$$\sum_p \omega_p(\tau) = 1, \quad (35)$$

which follows from definition (6). If one used inconsistent approximations for Σ_p and σ_p it could happened that this condition would be violated in some orders in g . We stress that if Σ_p and σ_p are determined self-consistently from single expression $\hat{\Sigma}$ this is not happening. Indeed, from Eq. (26) we can obtain following kinetic like equation

tion:

$$\begin{aligned} \frac{dW_p(\tau, \tau)}{d\tau} &= \int_0^\tau dt \Sigma_p(\tau - t) W_p(t, \tau) + \text{h.c.} \\ &+ \int_0^\tau dt \sigma_p(t, \tau) G_p(\tau - t) + \text{h.c.} \end{aligned} \quad (36)$$

It is more insightful to consider this equation in dual space. Namely, let us pick some quantity X_p and consider evolution of its average $\langle X \rangle \equiv \sum_p X_p \omega_p(\tau)$. Using this equation with self-energy parts given by (23) and (24) we obtain:

$$\frac{d\langle X \rangle}{d\tau} = \sum_{N=1}^{\infty} (ig)^{N+1} S_N(X) + \sum_{N=1}^{\infty} (-ig)^{N+1} S_N^*(X) \quad (37)$$

with

$$\begin{aligned} S_N(X) &= \sum_{p, |k|>1, |q|<1} (X_p - X_{p-k+q_1}) \int_{\Delta_\tau} \prod_{i=1}^{N+1} dt_i e^{-iE_{kq_1} t_1} \times \\ &\times G_{p-k+q_1}(t_1) \dots G_{p-k+q_N}(t_N) e^{-iE_{kq_N} t_N} W_p(t_{N+1}, \tau). \end{aligned} \quad (38)$$

This, in particular, shows that balance (35) is conserved for any moment of time (to see this one should put $X_k = 1$). The balance property (35) holds for any choice of $\hat{\Sigma}$.

In the next section we derive approximate expressions for the self-energy contributions and solve (29) and (30) in a way to ensure that sum rule (35) is satisfied up to order $O(g^2)$.

IV. ASYMPTOTIC DISTRIBUTION

A. Feynman Green Function

To find probability distribution function we first have to solve Eq. (29) to find out Feynman Green function. To do this we will first take the following expression in the leading order in g :

$$G_p(\lambda)^{-1} = \lambda + g^2 \sum_{|k|>1, |q|<1} \frac{1}{\lambda + iE_{kq}}. \quad (39)$$

This function has a cut in λ complex plane along imaginary axis. We can introduce spectral function A_p as:

$$G_p(\lambda) = \int dz \frac{A_p(z)}{\lambda + iz}, \quad (40)$$

then time dependence will be given as:

$$G_r(t) = \int d\omega A_r(\omega) e^{-i\omega t}. \quad (41)$$

From formula (39) one can easily conclude that:

$$A_p(\omega) = \frac{1}{\pi} \frac{\Gamma_p(\omega)/2}{(\omega - S_p(\omega))^2 + (\Gamma_p(\omega)/2)^2}, \quad (42)$$

where

$$\Gamma_p(\omega) = 2\pi\gamma^2 \left(\frac{2\pi}{L}\right)^2 \sum_{|k|>1, |q|<1} \delta(\omega - E_{kq}), \quad (43)$$

and

$$S_p(\omega) = \text{v.p.} \int \frac{dE}{2\pi} \frac{\Gamma_p(E)}{\omega - E}. \quad (44)$$

Remind also that $\gamma = gL/(2\pi)$. Function $\Gamma_p(\omega)$ is zero below some threshold. For $|p| > q_0 \equiv \min(1, \eta)$ this threshold is negative so $A_p(\omega)$ has a shape of Lorentz distribution centered approximately at $\omega = 0$ with width $\Gamma_p \equiv \Gamma_p(0)$. Such form of spectral function is typical for decay processes that in our case reflect the possibility for impurity to scatter on host particle and lose momentum. Inverse width determines time-scale prior to which $G_p(t)$ has diffusive dynamic. Namely, for:

$$1 \ll t \lesssim \frac{1}{\gamma^2} \log \frac{1}{\gamma^2}, \quad (45)$$

$$G_p(t) = e^{-\frac{\Gamma_p}{2} t}. \quad (46)$$

On the other hand, for $|p| < q_0$ main contribution comes from domain where $\Gamma_p = 0$, in such case we can replace spectral function to $A_p(\omega) = \delta(\omega - S_p(\omega))$, which gives some oscillatory contribution which is not important for our consideration for times satisfying (45), therefore, for $|p| < q_0$ we have:

$$G_p(t) = 1. \quad (47)$$

Even though this naive dynamics might acquire some subdiffusive corrections [28],[29],[30] they are important beyond time-scale (45). Here we would like to stress that we are not considering genuine $t \rightarrow \infty$ but just large enough time meaning that $e^{-\gamma^2 t} \sim \gamma^2$ which seems to be appropriate at $\gamma \rightarrow 0$ case.

The width Γ_p can be easily calculated from its definition (43), which is nothing but Fermi Golden Rule. So, for $p > q_0$ we have:

$$\frac{\Gamma_p}{2\pi\gamma^2} = \begin{cases} \theta(1-p) \log \frac{1+\eta}{1-\eta} + \theta(p-1) \log \frac{p+\eta}{p-\eta}, & \eta < 1 \\ \log \left| \frac{p+\eta}{p-\eta} \right| - \theta(\eta-p) \log \frac{\eta+1}{\eta-1}, & \eta > 1 \\ \log \frac{p+1}{p-1}, & \eta = 1 \end{cases} \quad (48)$$

Time-domain (45) in λ space can be easily expressed as

$$\frac{\gamma^2}{\log \gamma^{-2}} \lesssim |\lambda| \ll 1 \quad (49)$$

which on practice means that for $\Sigma_p(\lambda)$ in (29) one can consider $\lambda \rightarrow 0$ keeping it only to regularize possible divergences in denominator. Performing calculations analogous to the Boltzmann case we see that ladder contributions do not change result (46) in the leading order. Moreover, for equal mass one can confirm result for spectral function directly from Bethe Ansatz [22]. The only places where higher orders in γ may play role are vicinities of the discontinuities of Fermi Golden Rule result (48), $p = 1$, $p = \eta$ and probably some other countable set of points (see [21, 31]). But such consideration is beyond the scope of this paper.

B. Generalized distribution function (Boltzmann case)

Once we have determined Green function as:

$$G_p(\lambda) = \frac{1}{\lambda + \Gamma_p/2}, \quad (50)$$

we can solve QBE (28). For the bubble diagrams (Fig. 1) QBE reads:

$$W_p(\lambda_1, \lambda_2) = \frac{\omega_p^{(0)}}{(\lambda_1 + \Gamma_p/2)(\lambda_2 + \Gamma_p/2)} + g^2 \sum_{|s| < 1, |k| > 1} \frac{W_{p-s+k}(\lambda_1 + iE_{sk}, \lambda_2 - iE_{sk})}{(\lambda_1 + \Gamma_p/2)(\lambda_2 + \Gamma_p/2)}. \quad (51)$$

The asymptotic distribution is given by the formula

$$w_p^\infty = W_p(\lambda_1, \lambda_2)^{\text{L.T.}} \quad (52)$$

where L.T. stands for Laplace transformation, namely

$$W_p(\lambda_1, \lambda_2)^{\text{L.T.}} \equiv \lim_{t \rightarrow \infty} \int_C \frac{d\lambda_1}{2\pi i} \int_C \frac{d\lambda_2}{2\pi i} e^{t(\lambda_1 + \lambda_2)} W_p(\lambda_1, \lambda_2) \quad (53)$$

One could easily conceive that each iteration will give just sum of the inverse polynomials in λ_i . The long time limit (53) means that only residues at $\lambda_i = 0$ are important. One can also show that this poles are simple, so we can ignore all other λ_i dependence by evaluating corresponding functions at $\lambda_i = 0$. Therefore, since $\Gamma_q = \Gamma_{p_0} > 0$ we can safely put $\lambda_1 = \lambda_2 = 0$ in the square brackets in (57). After that we perform summation over s and k . The only way to get non-vanishing result at $\gamma \rightarrow 0$ is to make zeroes of E_{sk} lie in the summation domain. This

Here C is the contour that goes from $-i\infty$ to $i\infty$ and lies to the right from all the singularities of the integrand, which in our case means $\text{Im}\lambda > 0$; $t \rightarrow \infty$ means the right edge of the time-domain (45). Initial distribution that we for simplicity assume to be $\omega_p^{(0)} = \delta_{p,p_0}$, can be easily extended to arbitrary diagonal due to linearity.

The structure of QBE immediately suggest that solution can be obtained by the iterative procedure. Because balance is conserved (35), it is reasonable to make iterations till

$$\sum_p \omega_p^\infty = 1 \quad (54)$$

in the limit $\gamma \rightarrow 0$. For example, if initial momentum $|p_0| < q_0$, then even "zero" iteration term will do the job. Namely,

$$W_p^{(0)}(\lambda_1, \lambda_2) = \frac{1}{\lambda_1 + \Gamma_p/2} \omega_p^{(0)} \frac{1}{\lambda_2 + \Gamma_p/2} \quad (55)$$

gives:

$$\omega_p^\infty = \omega_p^{(0)} \lim_{t \rightarrow \infty} e^{-t\Gamma_p}. \quad (56)$$

But for $p = p_0 < q_0$ we have $\Gamma_p = 0$ and condition (54) is saturated, providing $\omega_p^\infty = \omega_p^{(0)}$. This is merely shows kinematic impossibility of the single scattering due to Pauli blocking [21, 23].

Assume now that initial momentum $p_0 > q_0$ then $\Gamma_{p_0} > 0$ and contribution of the "zero" iteration term is negligible, while the first iteration gives:

$$W_p^{(1)}(\lambda_1, \lambda_2) = \frac{1}{\lambda_1 + \Gamma_p/2} \left[g^2 \sum_q \sum_{|s| < 1, |k| > 1} \frac{\omega_q^{(0)} \delta_{q,p-s+k} \theta(|q-p+s|-1)}{(\lambda_1 + iE_{sk} + \Gamma_q/2)(\lambda_2 - iE_{sk} + \Gamma_q/2)} \right] \frac{1}{\lambda_2 + \Gamma_p/2}. \quad (57)$$

will lead to restriction for momenta q and p to lie in a certain domain: $(q, p) \in \Omega$, which we specify below (c.f. [21, 31]). This way, we get:

$$W_p^{(1)}(\lambda_1, \lambda_2) = \frac{1}{\lambda_1 + \Gamma_p/2} \left(\sum_q \omega_q^{(0)} \mathcal{P}_{q \rightarrow p}^{(1)} \right) \frac{1}{\lambda_2 + \Gamma_p/2}, \quad (58)$$

with

$$\mathcal{P}_{q \rightarrow p}^{(1)} = \frac{2\pi}{L} 2\pi\gamma^2 \frac{\theta_\Omega(q, p)}{|p-q|\Gamma_q}. \quad (59)$$

Here the step function $\theta_\Omega(q, p)$ means that point (q, p) lies in the domain Ω . More specifically this means that:

$$\theta_\Omega(q, p) \equiv \theta \left(\left| q \frac{1+\eta}{2\eta} + p \frac{1-\eta}{2\eta} \right| - 1 \right) \theta \left(1 - \left| q \frac{1-\eta}{2\eta} + p \frac{1+\eta}{2\eta} \right| \right). \quad (60)$$

If initial momentum is such that all $|p|$ satisfying condition (60) are less than q_0 then this iteration gives final answer:

$$\omega_p^\infty = [\omega_p^\infty]^{(1)} = \theta(q_0 - |p|) \sum_q \omega_q^{(0)} \mathcal{P}_{q \rightarrow p}^{(1)} \quad (61)$$

This happens for $|p_0| < q_1$ with $q_1 = \frac{3\eta-1}{\eta+1}$ for $\eta > 1$ and $q_1 = \frac{\eta(3-\eta)}{\eta+1}$ for $\eta < 1$. Kinematically $[q_0, q_1]$ is the range of momenta within which impurity momentum drops below q_0 in one scattering [31]. In the general case we must reiterate till $|p| < q_0$. If this happens after n iteration then corresponding probability distribution reads:

$$\omega_p^\infty = \theta(q_0 - |k|) \sum_{j=1}^n \mathcal{P}_{p_0 \rightarrow k}^{(j)} \quad (62)$$

where

$$\mathcal{P}_{p_0 \rightarrow k}^{(j)} = \sum_{|q| > q_0} \mathcal{P}_{p_0 \rightarrow q}^{(1)} \mathcal{P}_{q \rightarrow k}^{(j-1)}. \quad (63)$$

If the initial momentum is below $q_\infty = \max(1, \eta)$ then we need finite number of iterations, if not, the exact answer is given only by infinite number of iterations, though approximation error is very well controlled [31].

In the case of equal masses it turns out that first iteration gives full (valid for any initial momentum) asymptotic solution of Eq. (61), which reads:

$$\omega_p^\infty = \frac{2\pi}{L} \frac{1}{\log \frac{p_0+1}{p_0-1}} \frac{\theta(1-|p|)}{p_0-p}, \quad p_0 > 1. \quad (64)$$

We would like to emphasize that this result was obtained in the lowest possible approximation for the self-energy (Fig. 1) in the next subsection we show that ladder diagrams modify this result substantially even in the leading γ order.

Also let us comment on how to derive usual Boltzmann equation to describe not only asymptotic distributions but time dependence at the approaching of this asymptotic as well. Note that in time domain Eq. (51) takes the following form:

$$W_p(t_1, t_2) = \left(\omega_p^{(0)} + g^2 \sum_{|s| < 1, |k| > 1} \int_0^{t_1} d\tau_1 \int_0^{t_2} d\tau_2 e^{\frac{\Gamma_p \tau_1}{2}} \times \right. \\ \left. \times e^{-iE_{sk}(\tau_1 - \tau_2)} W_{p-s+k}(\tau_1, \tau_2) e^{\frac{\Gamma_p \tau_2}{2}} \right) e^{-\frac{(t_1+t_2)\Gamma_p}{2}} \quad (65)$$

From this equation we get:

$$\frac{d\omega_p(t)}{dt} \equiv \frac{dW_p(t, t)}{dt} = -\Gamma_p \omega_p(t) + \\ + g^2 \sum_{|s| < 1, |k| > 1} \int_0^t d\tau e^{-iE_{sk}\tau - \gamma^2 \Gamma_p \tau} W_{p-s+k}(t-\tau, t) \\ + g^2 \sum_{|s| < 1, |k| > 1} \int_0^t d\tau e^{iE_{sk}\tau - \gamma^2 \Gamma_p \tau} W_{p-s+k}(t, t-\tau). \quad (66)$$

Now assuming weak dependence of the $W_p(t_1, t_2)$ on the difference between t_1 and t_2 we may put $W_p(t, t-\tau) \approx W_p(t-\tau, t) \approx \omega_p(t)$, therefore equation (66) at $t \rightarrow \infty$ will take following form:

$$\frac{d\omega_p(t)}{dt} = -\Gamma_p \omega_p(t) + g^2 \sum_{|s| < 1, |k| > 1} 2\pi \delta(E_{sk}) \omega_{p-s+k}(t). \quad (67)$$

Furthermore, using notations (59), (60) we can present it as:

$$\frac{d\omega_p(t)}{dt} = -\Gamma_p \omega_p(t) + \sum_q \Gamma_{q \rightarrow p} \omega_q(t), \quad (68)$$

where

$$\Gamma_{q \rightarrow p} = \frac{2\pi}{L} 2\pi \gamma^2 \frac{\theta_\Omega(q, p)}{|p-q|}, \quad (69)$$

Obviously $\sum_q \Gamma_{p \rightarrow q} = \Gamma_p$. Boltzmann equation in the form (68) was the starting point in Ref. [31], where it was derived directly from Fermi Golden Rule treatment of the quantum mechanical transition probability.

C. Generalized distribution function (Ladder case)

In the previous subsection we have considered asymptotical solution based on the self-energy given by bubble diagram (Fig. 1). The ladder effects change QBE by introducing denominators as in (30). To estimate those effects we can check how the residue at the pole $\lambda_i = 0$ is affected by the presence of corresponding denominators. Then the first iteration result of Eq. (57) is modified by multiplication of the summand on the $|f_{p,s}|^2$, where:

$$f_{p,s} = \left(1 - g \sum_{|q| < 1} \frac{1}{E_{sq} - i0} \right)^{-1}. \quad (70)$$

Since $|s| < 1$ the expression in denominator will always have imaginary part:

$$f_{p,s} = \left(1 + (\dots) + i\pi g \sum_{|q| < 1} \delta(E_{sq}) \right)^{-1} = \\ = \left(1 + (\dots) + i2\pi\eta \frac{\gamma}{|\eta-1|} \frac{1 + \theta(1-|s-2p|)}{|p_0-p|} \right)^{-1} \quad (71)$$

here dots denote some irrelevant real part. We see that Boltzmann description (51) is valid and all results from the previous subsection are correct if

$$\frac{\gamma}{|\eta - 1|} \ll 1. \quad (72)$$

On the other hand, in the equal masses case ($\eta = 1$) the contribution of the residues at poles of Green functions

is negligible. This way, for $\eta = 1$ probability distribution function is completely determined by the residues at poles of $\sigma_p^L(\lambda_1, \lambda_2)$ (30) that come from ladder contributions. They are responsible for multiple scattering events.

From now on we focus on the equal masses case, $\eta = 1$. Analogously to the previous subsection let us consider initial momentum $p_0 > q_0 = 1$. After the first iteration, instead of (57) one can get:

$$W_p^{(1)}(\lambda_1, \lambda_2) = g^2 \sum_{|s| < 1, |k| > 1} \frac{G_p(\lambda_1) G_{p-s+k}(\lambda_1 + iE_{sk}) G_{p-s+k}(\lambda_2 - iE_{sk}) G_p^*(\lambda_2) \delta_{p-s+k, p_0}}{\left(1 - ig \sum_{|q| < 1} G_{p-s+q}(\lambda_1 + iE_{sq})\right) \left(1 + ig \sum_{|q| < 1} G_{p-s+q}^*(\lambda_2 - iE_{sq})\right)}, \quad (73)$$

here, for the sake of brevity, we use notation $G_p(\lambda)$ defined in Eq. (50). Now taking into account that final distribution is determined after Laplace transformation at equal times (53) we may shift $\lambda_1 \rightarrow \lambda_1 + i\xi$, $\lambda_2 \rightarrow \lambda_2 - i\xi$ for any real ξ . Moreover, we find it simpler not to con-

sider probability distribution function but use dual description of some quantity X_p and its average:

$$\langle X \rangle = \sum_p X_p \omega_p^\infty \quad (74)$$

Using formula (73) we obtain:

$$\langle X \rangle = g^2 \sum_{|s| < 1, |k| > 1} \left[\frac{X_{p_0+s-k} G_{p_0+s-k}(\lambda_1 + i(k-p_0)(k-s)) G_{p_0}(\lambda_1) G_{p_0}(\lambda_2) G_{p_0+s-k}^*(\lambda_2 - i(k-p_0)(k-s))}{\left(1 - ig \sum_{|q| < 1} G_{p_0-k+q}(\lambda_1 + i(k-p_0)(k-q))\right) \left(1 + ig \sum_{|q| < 1} G_{p_0-k+q}^*(\lambda_2 - i(k-p_0)(k-q))\right)} \right]^{L.T.} \quad (75)$$

Remind that L.T. means that in order to obtain $\langle X \rangle$ one has to perform Laplace transformation on both λ_1 and λ_2 at the same time, and then send this time to infinity, similar to Eq. (53). The dominator in sum in (75) as a function of λ has zero at

$$i\lambda = (k - p_0) \left(k - \coth \left(\frac{k - p_0}{2\gamma} \right) \right) \quad (76)$$

Calculating residues in this point we obtain from Eq. (75) the following answer:

$$\begin{aligned} \langle X \rangle &= \left(\frac{2\pi}{L} \right)^2 \sum_{\substack{|s| < 1 \\ |k| > 1}} \frac{X_{p_0+s-k}}{\left(s - \coth \frac{k-p_0}{2\gamma} \right)^2 \left(1 - \cosh \frac{k-p_0}{\gamma} \right)^2} \times \\ &\times \frac{(k - p_0)^2}{(k - p_0)^2 \left(k - \coth \frac{k-p_0}{2\gamma} \right)^2 + (\Gamma_{p_0}/2)^2} \quad (77) \end{aligned}$$

The integral over k acquires it's value from the small domain around k_* which is solution of the following equa-

tion:

$$k_* = \coth \frac{k_* - p_0}{2\gamma}, \quad (78)$$

which at $\gamma \rightarrow 0$ can be written as:

$$k_* = p_0 + 2\gamma \coth^{-1} p_0 = p_0 + \frac{\Gamma_{p_0}}{2\pi\gamma}. \quad (79)$$

Now expanding integrand near this value and assuming that X_s is smooth function of momentum we get:

$$\begin{aligned} \langle X \rangle &\stackrel{\gamma \rightarrow 0}{\approx} \left(\frac{2\pi}{L} \right)^2 \sum_{|s| < 1, |k| > 1} \frac{X_s}{(p_0 - s)^2} \frac{(p_0^2 - 1)^2}{4} \times \\ &\times \frac{\gamma^2}{(k - k_*)^2 (p_0^2 - 1)^2 / 4 + (\pi\gamma^2)^2} = \\ &= \frac{2\pi}{L} \frac{p_0^2 - 1}{2} \sum_{|s| < 1} \frac{X_s}{(p_0 - s)^2}. \quad (80) \end{aligned}$$

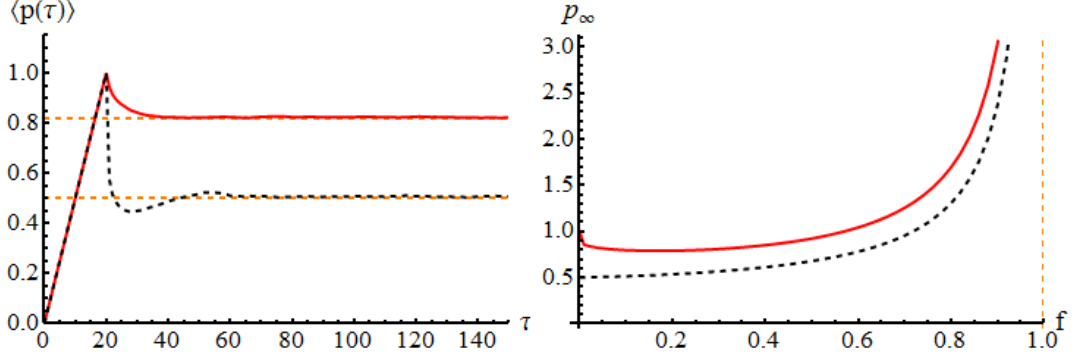


FIG. 3. Left panel: average momentum of the impurity v.s. time in equal masses case for the applied external force $f = 0.05$. Right panel: steady state momentum v.s. applied force obtained by formulas (90) (red solid line) and (91) (black dashed line). The steady state momentum diverges at a critical force $f_{c1} = 1$ (marked by a dashed vertical line). On both panels solid red lines corresponds to the correct (ladder) kinetic equation, while black dashed lines show results of the Fermi Golden Rule kinetic equation.

So comparing with the formula (74) we obtain

$$\omega_p^\infty = \frac{2\pi}{L} \frac{p_0^2 - 1}{2} \frac{\theta(1 - |p|)}{(p_0 - p)^2} \quad (81)$$

One can note how different this answer is from result (64) which disregards ladder effects. Therefore, we see that multiple scattering events of the impurity on host particles change probability distribution function even in the leading order. In contrast, Feynman Green function remains practically the same.

Performing calculations similar to the previous subsection we can obtain analog of Boltzmann equation (68):

$$\frac{d\omega_p(t)}{dt} = -\Gamma_p \omega_p(t) + \sum_q \tilde{\Gamma}_{q \rightarrow p} \omega_q(t), \quad (82)$$

where

$$\tilde{\Gamma}_{q \rightarrow p} = \frac{2\pi}{L} \Gamma_q \frac{q^2 - 1}{2} \frac{\theta(1 - |p|)}{(q - p)^2}, \quad (83)$$

Obviously $\sum_q \tilde{\Gamma}_{p \rightarrow q} = \Gamma_p$. So this equation looks like ordinary Boltzmann equation, the only difference is that transition rates $\tilde{\Gamma}_{p \rightarrow q}$ are no longer determined from the Fermi-Golden Rule (69), even though total width remains unchanged.

V. FORCE

In Ref. [31] authors considered how constant force applied to the impurity affects dynamic in the case of light $\eta < 1$ and heavy impurity $\eta > 1$. Now using correct Boltzmann equation (82) we can also investigate $\eta = 1$ case. One can generalize Eq. (82) to account for the constant force F acting upon the impurity:

$$\frac{\partial \omega_k(t)}{\partial t} + F \frac{\partial \omega_k(t)}{\partial k} = -\Gamma_p \omega_p(t) + \sum_q \tilde{\Gamma}_{q \rightarrow p} \omega_q(t) \quad (84)$$

Without loss of generality we assume $f > 0$. In this case Eq. (84) allows us to put $\omega_k = 0$ for $k < -1$. For the rest values of k we can introduce following notations:

$$\omega_k = \frac{2\pi}{L} \begin{cases} \omega(k), & k \in [-1, 1] \\ \chi(k), & k > 1 \end{cases}, \quad (85)$$

where we performed certain re-scalings. It is also convenient to re-scale time and force:

$$f = \frac{F}{2\pi\gamma^2}, \quad \tau = 2\pi\gamma^2 t. \quad (86)$$

Then Eq. (84) reads:

$$\begin{aligned} \frac{\partial \chi(k, \tau)}{\partial \tau} + f \frac{\partial \chi(k, \tau)}{\partial k} &= -\log \frac{k+1}{k-1} \chi(k, \tau), \\ \frac{\partial \omega(k, \tau)}{\partial \tau} + f \frac{\partial \omega(k, \tau)}{\partial k} &= \int_1^\infty dq \chi(q, \tau) \frac{q^2 - 1}{2(q - k)^2} \log \frac{q+1}{q-1}. \end{aligned} \quad (87)$$

The dynamics of the average momentum defined as

$$\langle p(\tau) \rangle = \int_{-1}^1 dk k \omega(k, \tau) + \int_1^\infty dk k \chi(k, \tau) \quad (88)$$

is shown in the left panel of Fig. 3. It is compared to the dynamics that comes from the Boltzmann equation with transition rates determined by Fermi Golden Rule (68). We see that in both cases system quickly comes to the steady state. However, the momenta obtained from the correct kinetic equation that takes into account ladder effects are larger. The steady state is characterized by the following distribution function:

$$\chi(k) = \exp \left(-\frac{1}{f} \left((k-1) \log \frac{k+1}{k-1} + 2 \log \frac{k+1}{2} \right) \right). \quad (89)$$

up to some normalization constant. This function gets finite normalization for $f < f_{c0} = 2$. This way, f_{c0} is a critical force for which steady state exists. Corresponding average momentum reads

$$p_{\infty}^L = \frac{\int_1^{\infty} dq \chi(q) \left(\frac{3q^2-1}{2} \log \frac{q+1}{q-1} - 2q \right)}{\int_1^{\infty} dq \chi(q) \left(q \log \frac{q+1}{q-1} - 1 \right)} = \frac{1 - \frac{2-3f}{f} \int_1^{\infty} dq q \chi(q)}{1 - \frac{1-f}{f} \int_1^{\infty} dq \chi(q)}. \quad (90)$$

Note that if one would consider bubble diagram only one would get the following answer:

$$p_{\infty}^B = \frac{\int_1^{\infty} dq q \chi(q)}{2 \int_1^{\infty} dq \chi(q)}. \quad (91)$$

All those expressions are finite for $f < f_{c1} = 1$. This two results are compared at Fig. 3. We see that calculations that take into account ladder contribution give higher steady state momentum. Unfortunately at small forces $f \ll \gamma^2$ our results are still hardly applicable because average momentum is equal to Fermi momentum where our simple approximation for Feynman Green function (50) is inapplicable and more careful solution of the Dyson equation needed. We hope to clarify this issue in future.

VI. SUMMARY AND DISCUSSION

To summarize, we have systematically derived kinetic equation of impurity in Tonks-Girardeau gas in weak-coupling regime. We have re-derived some of the results of [31] when masses of the host particle and impurity are different, confirmed applicability of the approach used there and generalized description to the equal mass case. Our Quantum Boltzmann Equation correctly takes into account multiple coherent scatterings and reproduce results that follows from Bethe-Ansatz treatment [21, 22]. At equal masses we have also derived Boltzmann like equation which is absolutely new. This allowed us to consider the constant external force applied to the impurity and calculate steady state momentum. We have showed that application of the naive Boltzmann equation with the Fermi Golden Rule transition rates gives completely wrong answer in the equal masses case.

Perturbative description of the system developed here, is a powerful and versatile approach. It allows straightforward generalization for the arbitrary weak interaction between host and particle and for quite general host, which is the subject of further investigations. Another important possible generalization is to consider the system at finite temperature and in arbitrary trap potential. We believe that our approach will be fruitful in these cases as well. From the theoretical point of view it is also interesting to consider next to leading order corrections. This looks the most challenging issue not only because one have to find exact solution of the QBE but also take into account other diagrams not considered here. Nevertheless, approach described in this manuscript allows to do this systematically which we believe will be done in the nearest future.

ACKNOWLEDGMENTS

The author is grateful to V. Cheianov, O. Lychkovskiy, E. Burovskiy, M. Zvonarev and L. Glazman and for fruitful discussions.

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- [1] I. M. Khalatnikov and V. N. Zharkov, “Mobilities of he ions in liquid helium,” J. Exptl. Theoret. Phys. **32**, 1108 (1957).
 - [2] Lothar Meyer and F. Reif, “Mobilities of he ions in liquid helium,” Phys. Rev. **110**, 279–280 (1958).
 - [3] N. V. Prokofev, “Diffusion of a heavy particle in a fermi-liquid theory,” International Journal of Modern Physics B **7**, 3327–3351 (1993).
 - [4] Immanuel Bloch, Jean Dalibard, and Wilhelm Zwerger, “Many-body physics with ultracold gases,” Reviews of Modern Physics **80**, 885 (2008).
 - [5] Jacob F Sherson, Christof Weitenberg, Manuel Endres, Marc Cheneau, Immanuel Bloch, and Stefan Kuhr, “Single-atom-resolved fluorescence imaging of an atomic mott insulator,” Nature **467**, 68–72 (2010).
 - [6] Christof Weitenberg, Manuel Endres, Jacob F Sherson, Marc Cheneau, Peter Schauß, Takeshi Fukuhara, Immanuel Bloch, and Stefan Kuhr, “Single-spin addressing in an atomic mott insulator,” Nature **471**, 319–324 (2011).
 - [7] Stefan Palzer, Christoph Zipkes, Carlo Sias, and Michael Köhl, “Quantum transport through a tonks-girardeau gas,” Physical review letters **103**, 150601 (2009).
 - [8] Nicolas Spethmann, Farina Kindermann, Shincy John, Claudia Weber, Dieter Meschede, and Artur Widera, “Dynamics of single neutral impurity atoms immersed in an ultracold gas,” Phys. Rev. Lett. **109**, 235301 (2012).
 - [9] J Catani, G Lamporesi, D Naik, M Gring, M Inguscio,

- F Minardi, A Kantian, and T Giamarchi, “Quantum dynamics of impurities in a one-dimensional bose gas,” *Physical Review A* **85**, 023623 (2012).
- [10] Takeshi Fukuhara, Adrian Kantian, Manuel Endres, Marc Cheneau, Peter Schauß, Sebastian Hild, David Bellem, Ulrich Schollwöck, Thierry Giamarchi, Christian Gross, *et al.*, “Quantum dynamics of a mobile spin impurity,” *Nature Physics* **9**, 235–241 (2013).
- [11] Takeshi Fukuhara, Peter Schauß, Manuel Endres, Sebastian Hild, Marc Cheneau, and Immanuel Bloch, “Microscopic observation of magnon bound states and their dynamics,” *Nature* **502**, 76–79 (2013).
- [12] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, 2003).
- [13] Adilet Imambekov, Thomas L. Schmidt, and Leonid I. Glazman, “One-dimensional quantum liquids: Beyond the luttinger liquid paradigm,” *Rev. Mod. Phys.* **84**, 1253–1306 (2012).
- [14] Joachim Brand Alexander Yu. Cherny, Jean-Sebastien Caux, “Theory of superfluidity and drag force in the one-dimensional bose gas,” *Front. Phys.* **7**(1), 54–71 (2012).
- [15] D. M. Gangardt and A. Kamenev, “Bloch oscillations in a one-dimensional spinor gas,” *Phys. Rev. Lett.* **102**, 070402 (2009).
- [16] M. Schecter, A. Kamenev, D. M. Gangardt, and A. Lamacraft, “Critical velocity of a mobile impurity in one-dimensional quantum liquids,” *Phys. Rev. Lett.* **108**, 207001 (2012).
- [17] M. Schecter, D. M. Gangardt, and A. Kamenev, “Dynamics and bloch oscillations of mobile impurities in one-dimensional quantum liquids,” *Annals of Physics* **327**, 639–670 (2012).
- [18] Charles J. M. Mathy, Mikhail B. Zvonarev, and Eugene Demler, “Quantum flutter of supersonic particles in one-dimensional quantum liquids,” *Nature Physics* **8**, 881–886 (2012).
- [19] Michael Knap, Charles J. M. Mathy, Mikhail B. Zvonarev, and Eugene Demler, “Quantum flutter versus bloch oscillations in one-dimensional quantum liquids out of equilibrium,” arXiv:1303.3583 (2013).
- [20] M Girardeau, *J. Math. Phys.* **1**, 516 (1960).
- [21] O. Gamayun O. Lychkovskiy E. Burovski, V. Cheianov, “Momentum relaxation of a mobile impurity in a one-dimensional quantum gas,” arXiv:1308.6147.
- [22] E. Burovski, V. Cheianov, O. Gamayun, and O. Lychkovskiy, to appear (The numerical code we used is available at <https://bitbucket.org/burovski/mcba>) (2014).
- [23] O Lychkovskiy, arXiv:1308.6260 (2013).
- [24] JB McGuire, “Interacting fermions in one dimension. i. repulsive potential,” *Journal of Mathematical Physics* **6**, 432 (1965).
- [25] Keldysh L. V., *Zh. Eksp. Teor. Fiz* **47**, 1515 [*Sov. Phys. JETP*, 1965, 20, 1018] (1964).
- [26] Alex Levchenko Alex Kamenev, *Advances in Physics* **58**, 197 (2009).
- [27] A. Altland and B. Simons, *Condensed matter field theory* (Cambridge University Press, 2010).
- [28] M. B. Zvonarev, V. V. Cheianov, and T. Giamarchi, “Spin dynamics in a one-dimensional ferromagnetic bose gas,” *Phys. Rev. Lett.* **99**, 240404 (2007).
- [29] A. Kamenev and L. I. Glazman, “Dynamics of a one-dimensional spinor bose liquid: A phenomenological approach,” *Phys. Rev. A* **80**, 011603 (2009).
- [30] M. Khodas, M. Pustilnik, A. Kamenev, and L. I. Glazman, “Fermi-luttinger liquid: Spectral function of interacting one-dimensional fermions,” *Phys. Rev. B* **76**, 155402 (2007).
- [31] O. Gamayun, O. Lychkovskiy, and V. Cheianov, “Kinetic theory for a mobile impurity in a degenerate tonks-girardeau gas,” arXiv:1402.6362 (2014).